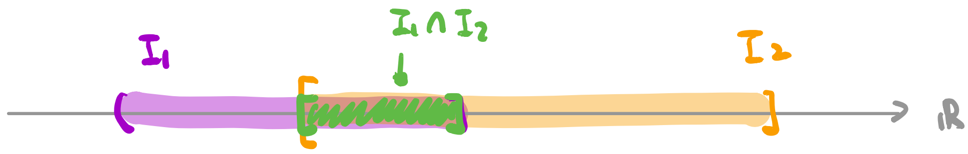
MATH 2050 C Lecture 6 (Jan 28)

[Problem Set 3 posted, due on Feb 5.]

Last time ...... interval, characterization by "connectedness".

Note: $I_1, I_2 \subseteq \mathbb{R}$ intervals $\Rightarrow$ $I_1 \cap I_2$ is always an interval.

BUT $I_1 \cup I_2$ might NOT be.



Q: What about $\bigcap_{i=1}^{\infty} I_i$?

Thm: ("Nested Interval Property" NIP)

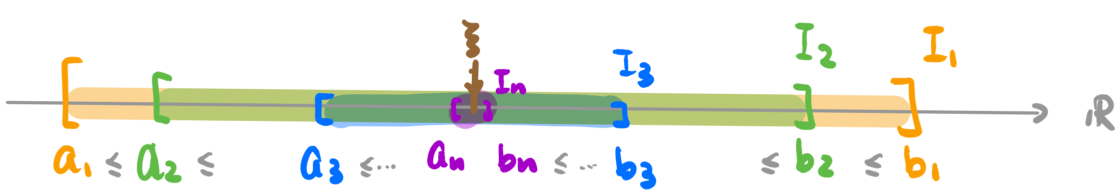
Let $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, be a seq. of closed and bounded intervals which are "nested":

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots\cdots\cdots$$

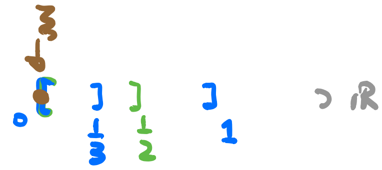
Then, $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Moreover, if $\inf\{\text{Length}(I_n) \mid n \in \mathbb{N}\} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Picture:



Examples: $\bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\}$



$$\bigcap_{n=1}^{\infty} [0, 1+\frac{1}{n}] = [0,1] \neq \phi .$$

Non-examples:

(1) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$ not closed!

(2) $\bigcap_{n=1}^{\infty} [n, \infty) = \phi$ not bdd!

(3) $\bigcap_{n=1}^{\infty} [n, n+1] = \phi$ not nested!

Proof of Thm:

Recall: $I_n = [a_n, b_n]$, where $a_n \leq b_n \quad \forall n \in \mathbb{N}$.

Nested $\Rightarrow$ $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1 \quad \forall n \in \mathbb{N}$

Consider $\phi \neq S := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

Note that S is bdd above since $a_n \leq b_1 \quad \forall n \in \mathbb{N}$.

By Completeness Property, $\xi := \sup S \in \mathbb{R}$ exists.

Claim: $\xi \in \bigcap_{n=1}^{\infty} I_n$ (hence $\bigcap_{n=1}^{\infty} I_n \neq \phi$).

Pf of Claim: Want: $\xi \in I_n \quad \forall n \in \mathbb{N}$, ie. $a_n \leq \xi \leq b_n$

- $\xi = \sup S$ is an upper bd. $\Rightarrow$ $\xi \geq a_n \quad \forall n \in \mathbb{N}$
- To see why $\xi \leq b_n \quad \forall n \in \mathbb{N}$, we argue by contradiction.

Suppose NOT, i.e. $\xi > b_m$ for some $m \in \mathbb{N}$

$\xi = \sup S \Rightarrow b_m$ is <u>NOT</u> an upper bd for S

$\Rightarrow \exists\, k \in \mathbb{N}$ s.t. $b_m < a_k$ Contradiction !

<u>Case 1</u>: $m < k \Rightarrow b_k \le b_m < a_k \le b_k$

<u>Case 2</u>: $m \ge k \Rightarrow b_m < a_k \le a_m$

For the rest of the theorem, leave as exercise. $\square$

<u>Cor</u>: $\mathbb{R}$ is uncountable.

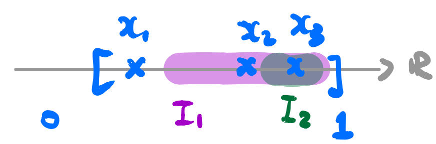
<u>Pf</u>: It suffices to show $[0,1]$ is uncountable.

Argue by contradiction. Suppose $[0,1]$ is <u>countable</u>.

Then we can list them all into a sequence:

$$[0,1] = \{x_1, x_2, x_3, x_4, \dots\dots\} \quad \cdots\cdots \quad (*)$$

Define a seq of nested, closed, bdd intervals $I_n, n \in \mathbb{N}$ as follow:



- choose $I_1 \subseteq [0,1]$ s.t. $x_1 \notin I_1$
- choose $I_2 \subseteq I_1$ s.t. $x_2 \notin I_2$

$\cdots\cdots\ \cdots$

- choose $I_n \subseteq I_{n-1}$ s.t. $x_n \notin I_n$

By NIP, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Suppose $\xi \in \bigcap_{n=1}^{\infty} I_n$.

$\Rightarrow \xi \in I_n \ \forall n \in \mathbb{N} \Rightarrow \xi \neq x_n \ \forall n \in \mathbb{N}$ Contradiction to $(*)$.

$\xi \in [0,1]$

$\square$